

INFLUENCE OF MODIFICATION OF BOUNDARY CONDITIONS ON LOAD CARRYING CAPACITY IN THIN-WALLED COLUMNS IN THE SECOND ORDER APPROXIMATION

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Abstract—The influence of modification of boundary conditions at both ends of the structures regarding interactive buckling on the postbuckling behaviour of thin-walled elastic columns with imperfections is studied. The investigation concerns thin-walled closed and open cross-section columns under axial compression. The columns are assumed to be simply supported at the ends. The asymptotic expansion established by Byskov and Hutchinson is employed in the numerical calculations in the form of the transition matrix method. The aim of the paper is to improve the study of the equilibrium path in the postbuckling behaviour of imperfect structures regarding the second order approximation.

NOMENCLATURE

a_{ijj}, b_{ijk}	three and four-index coefficients in the nonlinear equilibrium equations of eqn (17) (Byskov and Hutchinson, 1977)
b_i	width of the i th wall of the column
D_i	plate rigidity of the i th wall
E	Young's modulus
h_i	thickness of the i th wall of the column
I_{nnnn}	eqn (18)
J_{nnnn}	eqn (18)
l	length of the column
m	number of axial half-waves of n th mode
M_{ix}, M_{iy}, M_{ixy}	bending moment resultants for the i th wall
n	number of mode
N	number of interacting modes
N_{ix}, N_{iy}, N_{ixy}	in-plane stress resultants for the i th wall
$N_{ix}^{(nn)}, N_{iy}^{(nn)}, N_{ixy}^{(nn)}$	in-plane stress resultants for the i th wall in the second approximation
Q_{iy}^*	eqn (5)
u_i, v_i, w_i	displacement components of middle surface of the i th wall
u_i^0, v_i^0, w_i^0	prebuckling displacement fields
$u_i^{(n)}, v_i^{(n)}, w_i^{(n)}$	buckling displacement fields
$u_i^{(nn)}, v_i^{(nn)}, w_i^{(nn)}$	second order buckling displacement fields
Δ	measure of the applied pressure
λ	scalar load parameter
λ_n	value of λ at bifurcation mode number n
λ_s	maximum value of λ for imperfect column
ξ_n	amplitude buckling mode number n
ζ_n	imperfection amplitude corresponding to ξ_n
$\sigma_n^* = \sigma_n 10^3 / E$	dimensionless stress of mode number n
σ_m^*	min ($\sigma_1^*, \sigma_2^*, \sigma_3^*$)
σ_s^*	limit dimensionless stress for imperfect column (load carrying capacity).

1. INTRODUCTION

Thin-walled elements are widely used as structural components in many types of metal structures in which interaction buckling of elastic columns may result in an imperfection-sensitive structure and is the principal cause of collapse of thin-walled structures.

Contemporary load-bearing structures designed with the application of optimization methods contain a number of components subject to compression; the properties of such components are similar to those of ideal systems.

The concept of interactive buckling involves the general asymptotic theory of stability. Among all versions of the general nonlinear theory, the Koiter theory (Koiter, 1976) of conservative systems is most popular owing to its general character and development, even more so after Byskov and Hutchinson (1977) formulated it in a convenient way. The theory is based on asymptotic expansions of the postbuckling path and is capable of considering simultaneous or nearly simultaneous buckling modes. The expression for potential energy of the system expands in a series relative to the amplitudes of linear modes near the point of bifurcation; the latter generally corresponds to the minimum value of bifurcation loads.

For the first order approximation Koiter and van der Neut (1980) have proposed a technique in which the interaction of an overall mode with two local modes having the same wavelength has been considered. The fundamental mode is henceforth, called "primary" and the nontrivial higher mode (having the same wavelength as the "primary") corresponding to the mode triggered by overall longwave mode is called "secondary".

In the energy expression for the first order approximation the coefficients of the cubic terms $\xi_1\xi_2^2$, $\xi_1\xi_3^2$ and $\xi_1\xi_2\xi_3$ (where the index is 1 for the global mode, 2 for the primary local buckling mode and 3 for the secondary local mode) are the key terms governing the interaction. In the case of disregarding the interaction between the overall mode and the primary local mode and the secondary local mode, the coefficient of the $\xi_1\xi_2\xi_3$ term in the energy expression vanishes. In the analysis of a column with doubly symmetric cross-sections the coefficients of the $\xi_1\xi_2^2$ and $\xi_1\xi_3^2$ terms—the coefficients a_{ij} of the nonlinear system (17)—vanish.

If the analysis is restricted to the first approximation in solving the stability problem of the thin-walled structures the imperfection sensitivity can only be obtained. A determination of the postbuckling equilibrium path requires the second order approximation to be taken into account.

The structures where local buckling precedes the global one, $\lambda_1 \gg \lambda_2$ ($\sigma_1^* \gg \sigma_2^*$) are widely used. It is well known that structures with $\lambda_1 \gg \lambda_2$ can carry a load higher than that referring to the bifurcation value of local buckling. The behaviour of such structures cannot always be analysed in terms of the first nonlinear approximation where the limit load is always lower than the minimum value of the bifurcational load in the linear analysis. Therefore it is necessary to consider the second order approximation, that is the fourth order components in the potential energy (coefficients of the terms $\xi_i^2\xi_j^2$). In general, the stability analysis with regard to the second nonlinear approximation requires the solution of boundary value problems: for second order local, global and mixed modes [associated with the coefficients of the terms ξ_1^4 , ξ_i^4 and $\xi_1^2\xi_i^2$ (where $i > 1$) in the energy expression, respectively]. However, in the case when $\lambda_1 \gg \lambda_2$, the most significant are local second order modes. The second order global mode for Euler's model of the column is zero and in the case of an exact solution it is of little importance. The omission of the second order mixed mode involving the coefficients of the $\xi_1^2\xi_i^2$ ($i > 1$) and $\xi_i^2\xi_j^2$ ($i, j > 1$ and $i \neq j$) terms in the expression for potential energy is possible owing to the fact that the coefficients of the cubic terms $\xi_1\xi_2^2$, $\xi_1\xi_3^2$ and $\xi_1\xi_2\xi_3$ have already been included in the analysis. The admissibility of neglecting the mixed mode was shown by Koiter (1976), and Sridharan and Peng (1989).

The consideration of the coefficients of the terms $\xi_1\xi_2^2$, $\xi_1\xi_3^2$ and $\xi_1\xi_2\xi_3$ in the first nonlinear approximation involves taking into account the interaction between the global and two most dangerous local buckling modes having the same wavelength. This local secondary buckling mode is analogous to the mixed one; thus its consideration in the first approximation allows one to neglect the second order mixed modes (Koiter and van der Neut, 1980; Kolakowski, 1989a; Manevich, 1988; Pignataro and Luongo, 1987; Krolak, 1990; Sridharan and Ali, 1986; Sridharan and Peng, 1989).

In the present paper, which is a continuation of the paper by Kolakowski (1993) the analysis of the influence of modification of the boundary condition referring to the free support of the structure at both ends of the load carrying capacity is considered.

The analysis of the second nonlinear approximation considering only local second order modes (i.e. primary and secondary modes) was undertaken. The orthogonality condition for the first order and the local second order modes implies that it may be useful to change the amplitude of only three harmonics considered for each local second order mode. The

easiness of meeting the condition of orthogonality is one of the main reasons for choosing the method for solving the second order approximation in the form of series.

The equations for local modes in the second approximation depend not only on respective first order local modes, but, regarding the orthogonality conditions, also on the considered three first order modes. Therefore, none of the local second order modes obtained allowing for interactive buckling is identical with the mode obtained according to the theory of single-mode buckling (an uncoupled buckling), where the condition of orthogonality in relation to the global mode is not binding.

A more comprehensive review of the literature can be found in Kolakowski (1987a, 1989a, b, c), Manevich (1985, 1988), Moellmann and Goltermann (1989), Pignataro *et al.* (1985), Krolak (1990) and Sridharan and Ali (1985, 1986).

In the present paper the post-buckling behaviour of thin-walled structures in the elastic range under uniform compression is examined on the basis of Byskov and Hutchinson's method, with cooperation between all the walls of the structures being taken into account. The study is based on the numerical method of the transition matrix by Unger (1969), Klöppel and Bilstein (1971) and Bilstein (1974).

The most important advantage of this method is that it enables one to describe a complete range of behaviour of the thin-walled structures from global to local stability. In the solution obtained, the effects of interaction of certain modes having the same wavelength, the shear lag phenomenon and also the effect of cross-sectional distortions are included.

2. STRUCTURAL PROBLEM

The long thin-walled prismatic columns of length l , composed of plane, rectangular plate segments interconnected along longitudinal edges are considered.

The cross-section of this structure composed by several plates as well as the local Cartesian coordinate system are presented in Fig. 1.

The membrane strains of the i th wall are expressed by formulae :

$$\begin{aligned}\varepsilon_{ix} &= u_{i,x} + 0.5(w_{i,x}^2 + v_{i,x}^2), \\ \varepsilon_{iy} &= v_{i,y} + 0.5(w_{i,y}^2 + u_{i,y}^2), \\ 2\varepsilon_{ixy} &= \gamma_{ixy} = u_{i,y} + v_{i,x} + w_{i,x}w_{i,y},\end{aligned}\quad (1)$$

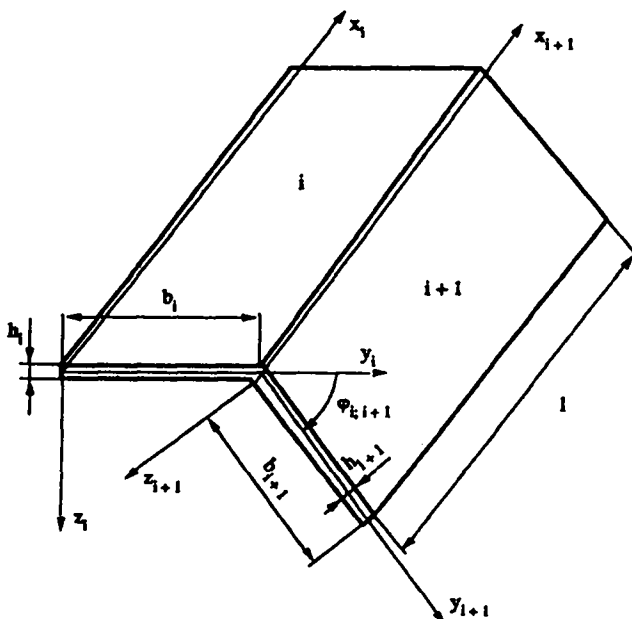


Fig. 1. Prismatic plate structure and the local coordinate system.

while the bending strains are given by :

$$\kappa_{ix} = -w_{i,xx} ; \quad \kappa_{iy} = -w_{i,yy} ; \quad \kappa_{ixy} = -w_{i,xy}. \quad (2)$$

The differential equilibrium equations resulting from the virtual work expression for a single wall can be formulated as follows :

$$\begin{aligned} -N_{ix,x} - N_{ixy,y} - (N_{iy}u_{iy})_{,y} &= 0, \\ -N_{iy,y} - N_{ixy,x} - (N_{ix}v_{ix})_{,x} &= 0, \\ D_i \nabla \nabla w_i - (N_{ix}w_{i,x})_{,x} - (N_{iy}w_{i,y})_{,y} - (N_{ixy}w_{i,x})_{,y} - (N_{ixy}w_{i,y})_{,x} &= 0. \end{aligned} \quad (3)$$

The geometrical and statical continuity conditions at the junctions of plates may be written in the form :

$$\begin{aligned} u_{(i+1)}|^- &= u_i|^+, \\ w_{(i+1)}|^- &= w_i|^+ \cos \varphi - v_i|^+ \sin \varphi, \\ v_{(i+1)}|^- &= w_i|^+ \sin \varphi + v_i|^+ \cos \varphi, \\ w_{(i+1),x}|^- &= w_{i,x}|^+, \\ M_{(i+1),y}|^- - M_{iy}|^+ &= 0, \\ N_{(i+1),y}|^- - N_{iy}|^+ \cos \varphi - Q_{iy}^*|^+ \sin \varphi &= 0, \\ Q_{(i+1),y}|^- + N_{iy}|^+ \sin \varphi - Q_{iy}^*|^+ \cos \varphi &= 0, \\ N_{ixy}|^- + N_{iy}u_{iy}|^- - N_{(i+1),xy}|^+ - N_{(i+1),y}u_{(i+1),x}|^+ &= 0, \end{aligned} \quad (4)$$

where Q_{iy}^* is an equivalent Kirchhoff's transverse force

$$\begin{aligned} Q_{iy}^* &= N_{iy}w_{i,y} + N_{ixy}w_{i,x} - D_i[w_{i,yy} + (2-\nu)w_{i,xy}], \\ \varphi &\equiv \varphi_{i,i+1}. \end{aligned} \quad (5)$$

The omission of the displacements of the fundamental state implies that we ignore the difference between configuration of the undeformed state and the fundamental state, and we may consequently regard the previously defined displacements u_i^0, v_i^0 as additional ones from the fundamental state to the adjacent state.

The prebuckling solution consists of homogeneous fields which are assumed as :

$$\begin{aligned} u_i^0 &= \Delta(l/2 - x_i), \\ v_i^0 &= \nu y_i \Delta, \\ w_i^0 &= 0. \end{aligned} \quad (6)$$

In the paper by Kolakowski (1993) the boundary conditions referring to the free support of the structure at both ends were assumed to be :

$$\begin{aligned} N_{ix}(x_i = 0, y_i) &= N_{ix}(x_i = l, y_i) = N_{ix}^0 = -Eh_i \Delta, \\ v_i(x_i = 0, y_i) &= v_i(x_i = l, y_i) = 0, \\ w_i(x_i = 0, y_i) &= w_i(x_i = l, y_i) = 0, \\ w_{i,xx}(x_i = 0, y_i) &= w_{i,xx}(x_i = l, y_i) = 0. \end{aligned} \quad (7a)$$

Such an analysis will be henceforth referred to as Analysis I.

In the present paper the boundary conditions at the ends are modified to be:

$$\begin{aligned} \frac{1}{b_i} \int N_{ix}(x_i = 0, y_i) dy_i &= \frac{1}{b_i} \int N_{ix}(x_i = l, y_i) dy_i = N_{ix}^0, \\ v_i(x_i = 0, y_i) &= v_i(x_i = l, y_i) = 0, \\ w_i(x_i = 0, y_i) &= w_i(x_i = l, y_i) = 0, \\ w_{i,xx}(x_i = 0, y_i) &= w_{i,xx}(x_i = l, y_i) = 0. \end{aligned} \quad (7b)$$

Such an analysis will henceforth be referred to as Analysis II.

The boundary conditions (7a) and (7b) permit the first order solution to be written as:

$$\begin{aligned} w_i^{(n)} &= W_i^{(n)}(y) \sin \frac{m\pi x}{l}, \\ u_i^{(n)} &= U_i^{(n)}(y) \cos \frac{m\pi x}{l}, \\ v_i^{(n)} &= V_i^{(n)}(y) \sin \frac{m\pi x}{l}, \end{aligned} \quad (8)$$

where $U_i^{(n)}(y)$, $V_i^{(n)}(y)$, $W_i^{(n)}(y)$ (with the m th harmonic) are initially unknown functions defined by the modified method of the transition matrix. The restraint conditions on the unloaded longitudinal edges of the adjacent plates are determined by applying the variational principle. The system of the differential equilibrium equations (3) is solved by the modified reduction method in which the state vector of the final edge is derived from the state vector of the initial edge by numerical integration of the differential equations in the y -direction using the Runge-Kutta formulae.

The assumed displacement field ensures compatibility of the corner displacements of the constituent plates. Thus $v = 0$ at the ends, implying that the plates are restrained in their plane at the ends. The point of major interest is that it can be employed to study the effect on the postbuckling stiffness of enforcing the compatibility of the displacements in the cross-sectional plane at the corners [for a more detailed analysis see the paper by Sridharan and Graves-Smith (1981)].

The global buckling mode occurs at $m = 1$ and the local modes at $m > 1$ (with $b_i \ll l$). Each buckling mode is normalized so that the maximum normal displacement is equal to the thickness of the first constituent wall.

The adopted boundary conditions (7a) and (7b) require an introduction of new functions for the second order fields

$$\begin{aligned} a_i^{(nn)} &= v_{i,y}^{(nn)} + v u_{i,x}^{(nn)}, \\ b_i^{(nn)} &= 0.5(1 - \nu)(u_{i,y}^{(nn)} + v_{i,x}^{(nn)}), \\ c_i^{(nn)} &= u_{i,x}^{(nn)} + \nu v_{i,y}^{(nn)}, \\ d_i^{(nn)} &= 0.5(1 - \nu)(u_{i,y}^{(nn)} - v_{i,x}^{(nn)}). \end{aligned} \quad (9)$$

This enables one to bring the boundary conditions (7a) and (7b), respectively, to a uniform condition by substituting

$$\bar{c}_i^{(nn)} = c_i^{(nn)} + \hat{c}_i^{(nn)} \quad (10)$$

where:

for Analysis I

$$\hat{c}_i^{(nm)} \equiv \hat{c}_i^{(nm)}(y) = 0.5 \left\{ \left(\frac{m\pi}{l} \right)^2 [(W_i^{(n)})^2 + (V_i^{(n)})^2] + v(U_{i,y}^{(n)})^2 \right\}, \tag{11a}$$

for Analysis II

$$\hat{c}_i^{(nm)} = \frac{1}{2b_i} \int_0^{b_i} \left\{ \left(\frac{m\pi}{l} \right)^2 [(W_i^{(n)})^2 + (V_i^{(n)})^2] + v(U_{i,y}^{(n)})^2 \right\} dy_i. \tag{11b}$$

Owing to the correction factor $\hat{c}_i^{(nm)}$ introduced into (10), the additional longitudinal compression caused by the influence of the first order displacement field upon the second order approximation can be reduced to zero at both ends. The above factor allows one to satisfy the boundary conditions (7a) and (7b), respectively, for the second order approximation. The problem will be treated more extensively in the further part of this paper.

It is easy to notice that various boundary conditions (7a) and (7b) cause only different values of the load carrying capacity for the second order approximation.

The functions defined by (9) and the correction factor (10), regarding the conditions :

$$u_{i,xy}^{(nn)} = u_{i,yx}^{(nn)} ; \quad v_{i,xy}^{(nn)} = v_{i,yx}^{(nn)} \tag{12}$$

enable one to bring the two first differential equations (3) in the second approximation to the system of four nonhomogeneous differential equations relative to $a_i^{(nn)}$, $b_i^{(nn)}$, $\bar{c}_i^{(nn)}$ and $d_i^{(nn)}$. The equation (3c) remains unchanged.

Regarding the above, the system of differential equations (3) for the second approximation may be written in the form :

for Analysis I

$$\begin{aligned} \bar{c}_{i,x}^{(nn)} + b_{i,y}^{(nn)} &= q_1 \\ a_{i,y}^{(nn)} + b_{i,x}^{(nn)} + \lambda N_{ix}^0 (b_{i,x}^{(nn)} - d_{i,x}^{(nn)}) / (1 - \nu) &= q_2 \\ (\bar{c}_{i,y}^{(nn)} - \hat{c}_i^{(nn)} - \nu a_{i,y}^{(nn)}) / (1 + \nu) - (b_{i,x}^{(nn)} + d_{i,x}^{(nn)}) &= 0 \\ (a_{i,x}^{(nn)} - \nu \bar{c}_{i,x}^{(nn)}) / (1 + \nu) - (b_{i,y}^{(nn)} - d_{i,y}^{(nn)}) &= 0 \\ D_i \nabla \nabla w_i^{(nn)} - E h_i / (1 - \nu^2) \lambda (u_{i,x}^0 + \nu v_{i,y}^0) w_{i,xx}^{(nn)} &= q_3 ; \end{aligned} \tag{13a}$$

for Analysis II

$$\begin{aligned} \bar{c}_{i,x}^{(nn)} + b_{i,y}^{(nn)} &= q_1 \\ a_{i,y}^{(nn)} + b_{i,x}^{(nn)} + \lambda N_{ix}^0 (b_{i,x}^{(nn)} - d_{i,x}^{(nn)}) / (1 - \nu) &= q_2 \\ (\bar{c}_{i,y}^{(nn)} - \nu a_{i,y}^{(nn)}) / (1 + \nu) - (b_{i,x}^{(nn)} + d_{i,x}^{(nn)}) &= 0 \\ (a_{i,x}^{(nn)} - \nu \bar{c}_{i,x}^{(nn)}) / (1 + \nu) - (b_{i,y}^{(nn)} - d_{i,y}^{(nn)}) &= 0 \\ D_i \nabla \nabla w_i^{(nn)} - E h_i / (1 - \nu^2) \lambda (u_{i,x}^0 + \nu v_{i,y}^0) w_{i,xx}^{(nn)} &= q_3 ; \end{aligned} \tag{13b}$$

where :

$$\begin{aligned} q_1 &= - (v_{i,x}^{(n)} v_{i,xx}^{(n)} + 2 \nu u_{i,y}^{(n)} u_{i,xy}^{(n)} + v_{i,y}^{(n)} u_{i,y}^{(n)} + v_{i,y}^{(n)} u_{i,y}^{(n)}) \\ &\quad + \nu u_{i,x}^{(n)} u_{i,yy}^{(n)} + w_{i,x}^{(n)} w_{i,xx}^{(n)} + 0.5 (1 + \nu) w_{i,y}^{(n)} w_{i,xy}^{(n)} \\ &\quad + 0.5 (1 - \nu) w_{i,x}^{(n)} w_{i,yy}^{(n)}) \\ q_2 &= - (u_{i,y}^{(n)} u_{i,yy}^{(n)} + 2 \nu v_{i,x}^{(n)} v_{i,xy}^{(n)} + u_{i,xx}^{(n)} v_{i,x}^{(n)} + u_{i,x}^{(n)} v_{i,xx}^{(n)}) \\ &\quad + \nu v_{i,y}^{(n)} v_{i,xx}^{(n)} + w_{i,y}^{(n)} w_{i,yy}^{(n)} + 0.5 (1 + \nu) w_{i,x}^{(n)} w_{i,xy}^{(n)} \\ &\quad + 0.5 (1 - \nu) w_{i,y}^{(n)} w_{i,xx}^{(n)}) \end{aligned}$$

$$\begin{aligned}
 q_3 = Eh_i / (1 - \nu^2) \{ & u_{i,xx}^{(n)} w_{i,x}^{(n)} + \nu v_{i,xy}^{(n)} w_{i,x}^{(n)} + u_{i,x}^{(n)} w_{i,xx}^{(n)} \\
 & + \nu v_{i,y}^{(n)} w_{i,xx}^{(n)} + v_{i,y}^{(n)} w_{i,y}^{(n)} + \nu u_{i,xy}^{(n)} w_{i,y}^{(n)} + v_{i,y}^{(n)} w_{i,yy}^{(n)} \\
 & + \nu u_{i,x}^{(n)} w_{i,yy}^{(n)} + 0.5(1 - \nu)(u_{i,y}^{(n)} w_{i,x}^{(n)} + v_{i,xy}^{(n)} w_{i,x}^{(n)} \\
 & + u_{i,xy}^{(n)} w_{i,y}^{(n)} + v_{i,xx}^{(n)} w_{i,y}^{(n)} + 2u_{i,y}^{(n)} w_{i,xy}^{(n)} + 2v_{i,x}^{(n)} w_{i,xy}^{(n)}) \}. \tag{14}
 \end{aligned}$$

The easiness of satisfying the orthogonality conditions of the second order stress field may be formulated as follows:

$$\begin{aligned}
 a_i^{(nn)} &= \sum A_{ik}^{(nn)}(y) \sin \frac{k\pi x}{l}, \\
 b_i^{(nn)} &= \sum B_{ik}^{(nn)}(y) \cos \frac{k\pi x}{l}, \\
 \bar{c}_i^{(nn)} &= \sum \bar{C}_{ik}^{(nn)}(y) \sin \frac{k\pi x}{l}, \\
 d_i^{(nn)} &= \sum D_{ik}^{(nn)}(y) \cos \frac{k\pi x}{l}, \\
 w_i^{(nn)} &= \sum W_{ik}^{(nn)}(y) \sin \frac{k\pi x}{l}. \tag{15}
 \end{aligned}$$

It should be noted that the second order field expressed by these formulae fulfils, at both ends of the structure, the condition

$$v_i^{(nn)}(x_i = 0, y_i) = 0; \quad v_i^{(nn)}(x_i = l, y_i) = 0. \tag{16}$$

$A_{ik}^{(nn)}, B_{ik}^{(nn)}, \bar{C}_{ik}^{(nn)}, D_{ik}^{(nn)}, W_{ik}^{(nn)}$ are unknown functions that shall be determined by the method of transition matrix as in the case of the first order approximation. The systems of differential equations for the second approximation are solved by the same modified reduction method; however, in this case the orthogonalization method by Bidermann (1977) is applied.

Poisson's effect is, as in the case of the first approximation, neglected at both ends and is taken into account inside the plate areas.

At the point where the load parameter λ_s reaches its maximum value for the imperfect structure (secondary bifurcation or limit points) the Jacobian of the nonlinear system of equations (Byskov and Hutchinson, 1977):

$$\left(1 - \frac{\lambda}{\lambda_j}\right) \xi_j + a_{ijj} \xi_i \xi_j + b_{ijk} \xi_i \xi_j \xi_k + \dots = \frac{\lambda}{\lambda_j} \bar{\xi}_j \quad \text{at } JK = 1, 2, \dots, N \tag{17}$$

is equal to zero.

Expressions for a_{ijj} and b_{ijk} are given in the Appendix. The formulae for the post-buckling coefficients a_{ijj} depend only on the buckling modes whereas coefficients b_{ijk} also depend on the second order field.

The result of integration along x indicates that postbuckling coefficients a_{ijj} are zero when the sum of the wave numbers associated with the three modes ($m_i + m_j + m_k$) is even, while b_{ijk} coefficients are zero if ($m_i + m_j + m_k + m_l$) is odd.

3. ANALYSIS OF RESULTS

A computer program was prepared in order to carry out numerical calculations of dependencies presented above. High accuracy was ensured in the calculation of basic quantities (four–five significant figures).

The numerical analysis concerning the second order fields in thin-walled structures depends on coefficients

$$\begin{aligned} b_{nnnn} &= [2\sigma^{(nn)} \cdot I_2(U^{(n)}) + 4\sigma^{(n)} \cdot I_{11}(U^{(n)}, U^{(nn)})] / (2\sigma^{(n)} \cdot \varepsilon^{(n)}) \\ &= (2J_{nnnn} + 4I_{nnnn}) / (2\sigma^{(n)} \cdot \varepsilon^{(n)}) \quad \text{where } n = 2, 3. \end{aligned} \quad (18)$$

determining the nature of postbuckling behaviour for uncoupled local buckling modes as functions of structural parameters and the influence of these coefficients upon equilibrium branches and the load carrying capacity.

The accuracy of trigonometric series convergence was found to be sufficient for practical purposes with up to 49 nonzero harmonics considered ($k = 1, 3, 5, \dots, 99$). This ensured obtaining the accuracy of the stress field below 0.1%.

In all analysed cases the postbuckling coefficients b_{2222} are positive. In most of them b_{3333} are positive unless it has been stated in the discussion of results.

In regard to “exact” continuity conditions at longitudinal sides of the structure for $\varphi_{i,i+1} \neq 0$ as early as in the first order approximation results in the appearance of nonzero displacement components, namely $u_i^{(n)} \neq 0$, $v_i^{(n)} \neq 0$, $w_i^{(n)} \neq 0$, hence in the second order approximation I_{nnnn} is not equal to zero. If the simplified boundary conditions are adopted for longitudinal sides, the above component is very often omitted.

The character of the second order field determining the postbuckling behaviour for uncoupled local buckling modes varies, subject to the mode of the first order buckling in question. The second order stresses are the sums of two components $\sigma^{(nn)} = \sigma_{11}^{(nn)} + \sigma_{22}^{(nn)}$, where: $\sigma_{11}^{(nn)}$ stands for stress corresponding to the first order displacements [this is caused by a quadratic operator in geometrical relations $\varepsilon = I_1(U) + 0.5I_2(U)$], $\sigma_{22}^{(nn)}$ represents stresses found on the basis of the second order displacement field [for more detailed analysis see the paper by Kolakowski (1993)].

The orthogonality condition of the global and local first order modes relative to the local second order modes implies that in the latter case amplitudes should be changed for $k = 1$ and $k = m$, if m is an odd number of half-waves in the longitudinal direction. This results from the adoption of series form (15) for the second order field and from the fact that the summation is carried out for odd k values.

The largest contribution to the second order fields is made by harmonics for low values of k ($k = 1, 3, 5$) and for k s close to $2m$. Hence amplitude corrections at $k = 1$ and $k = m$ (where $m = 3, 5, 7$) [which is characteristic for example, of open-section columns, compressed stiffened plates (Ali and Sridharan, 1988; Kolakowski, 1988, 1989a; Manevich, 1988; Krolak, 1990; Sridharan and Ali, 1986)] may lead to a significant change in the value of b_{nnnn} ($n = 2, 3$) coefficients.

The second order field was determined along with coefficients corresponding to the second order approximation b_{nnnn} for various values of λ (noted as λ_p); then the limit load capacity λ_s was found as a function of λ_p . Further it was analysed how imperfections affect the load carrying capacity λ_s .

When the load parameter λ comes close to the bifurcational value, the numerical conditioning of the system of equations (3) dependent on the determinant value approaching zero becomes worse. Since the second order stress field was assumed to be in series form (15) in order to avoid wrong numerical conditioning of equations, λ_p was taken as being equal to the lower of the two values: the limit load capacity $\lambda_s(\sigma_s^*)$ or $\lambda_m(\sigma_m^*)$ [as in the paper by Ali and Sridharan (1988)]. The stability of results (load carrying capacity λ_s) obtained for different λ_p and for $\lambda_p = \lambda_s$ (when $\lambda_s < \lambda_m$, which is possible only by using the method of successive approximations) was evaluated by the magnitude of mean relative error that should not exceed 3%. In most analysed cases a slight influence was found of the λ_p value upon the postbuckling coefficients b_{nnnn} .

More detailed consideration should also be given to the correction factors $\hat{c}_i^{(nm)}(y)$ (11a) and $\hat{c}_i^{(nm)}$ (11b), respectively. The thorough analysis was carried out on an evenly compressed square plate supported freely by its whole circumference, assuming that the longitudinal unloaded edges satisfy the condition $N_y = 0$, and $N_{xy} = 0$.

If the factor $\hat{c}_i^{(nm)}$ has been omitted in the analysis for any cross-section of the plate, the second order approximation gives an unbalanced force [that is, $\int N_{1x}^{(nm)} dy \neq 0$] and a post-buckling coefficient $b_{1111} = 1.0995$. It is known from the literature [for example by Klöppel *et al.* (1966)] that under such boundary conditions at longitudinal edges the value of this coefficient is 0.23551. In the case of taking into account the correction factor (11a), $\hat{c}_i^{(nm)}(y)$, the obtained value of b_{1111} was 0.2893, while in the case of a compressed plate strip $b_{1111} = 0.248$ for $\beta = l/b_1 = 3.0$ and $b_{1111} = 0.2399$ for $\beta = 5.0$. In the case of factor (11b), $\hat{c}_i^{(nm)}$, the values of b_{1111} were 0.367; 0.272 and 0.254 for $\beta = l/b_1 = 1.0$; 3.0 and 5.0, respectively.

The detailed numerical calculations presented below were carried out for columns whose cross-sectional geometry was analysed by Kolakowski (1989a).

3.1. Closed column

Geometrical dimensions of the cross-section of a thin-walled column subject to compression are presented in Fig. 2.

Table 1 presents nondimensional bifurcational stresses (linear problem) for a square column whose geometrical dimensions are:

$$b_1/b_2 = 1.0, \quad b_3/b_2 = 1.0, \quad h_1/h_2 = 1.0, \quad h_3/h_2 = 1.0, \\ l/b_2 = 67.39, \quad b_2/h_2 = 100.0, \quad \nu = 0.3,$$

as well as the ratio of limit stress to the minimum value of bifurcational stress for the first σ_{s1}^*/σ_m^* , and second σ_{s2}^*/σ_m^* nonlinear approximations, considering the interaction of three buckling modes (index n is 1 for the global mode, 2 for the primary local mode and 3 for the secondary local mode having the same number of half-waves as the primary one) and

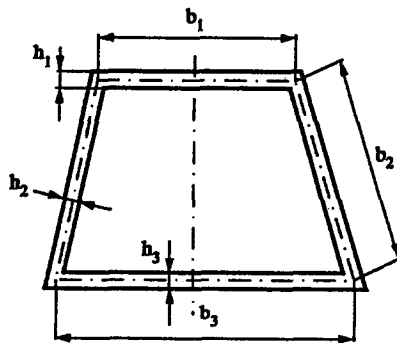


Fig. 2. Closed column geometry.

Table 1. Load carrying capacity for the square column of the following cross-section dimensions:

$$b_1/b_2 = b_3/b_2 = 1.0, \quad h_1/h_2 = h_3/h_2 = 1.0, \quad l/b_2 = 67.39, \quad b_2/h_2 = 100.0 \\ \text{at imperfections } |\xi_1| = 1.0, \quad |\xi_2| = 0.2, \quad \xi_3 = 0.0$$

m	Linear analysis			Nonlinear analysis		
	σ_1^*	σ_2^*	σ_3^*	σ_{s1}^*/σ_m^*	Analysis I σ_{s2}^*/σ_m^*	Analysis II σ_{s2}^*/σ_m^*
67	0.3614	0.3614	0.5199	0.7213	0.7433	0.7426
80	0.3614	0.3722	0.4904	0.7051	0.7258	0.7253

Table 2. Load carrying capacity for the trapezoidal column of the following cross-section dimensions :

$$b_1/b_2 = 0.5237, \quad b_3/b_2 = 1.0474, \quad h_1/h_2 = 0.4651, \\ h_3/h_2 = 1.5814, \quad l/b_2 = 46.095, \quad b_2/h_2 = 88.8 \\ \text{at imperfections } |\xi_1| = 1.0, |\xi_2| = 0.2, \xi_3 = 0.0$$

m	Linear analysis			Nonlinear analysis		
	σ_1^*	σ_2^*	σ_3^*	$\sigma_{12}^* \sigma_m^*$	Analysis I $\sigma_{12}^* \sigma_m^*$	Analysis II $\sigma_{12}^* \sigma_m^*$
54	0.5678	0.5659	1.1929	0.7654	0.8083	0.8263
69	0.5678	0.5983	0.9578	0.7485	0.7625	0.7756

the influence of numbers of half-waves m upon limit load capacity, the assumed imperfections being $|\xi_1| = 1.0$, $|\xi_2| = 0.2$, $\xi_3 = 0.0$. At $m = 80$ the limit load capacity σ_{12}^* reaches its minimum value for the assumed level of imperfections [for a more detailed analysis see Manevich (1982), Kolakowski (1987a, b, 1989a, c)]. The limit load capacity σ_{12}^* , nearly identical for Analysis I and II, was obtained.

Table 2 shows similar calculation results obtained for a trapezoid-section column :

$$b_1/b_2 = 0.5237, \quad b_3/b_2 = 1.0474, \quad h_1/h_2 = 0.4651, \quad h_3/h_2 = 1.5814, \\ l/b_2 = 46.095, \quad b_2/h_2 = 88.8, \quad \nu = 0.3$$

with identical imperfection values.

It should be noted that geometrical dimensions of the cross-section were selected so that the moments of inertia of the section relative to central axes of inertia could be identical (Kolakowski, 1989a, c).

In the case $m = 69$ a minimum value of the limit load is reached analogous to the case of a two-fold axis of symmetry ($m = 80$, Table 1). In both cases they are obtained regarding the influence of imperfections and the number of m [see examples by Kolakowski (1987a, b, 1989a)].

The minimum value of σ_{12}^* was obtained for Analysis I. As the values of the stress σ_1^* and σ_2^* are nearly simultaneous (Tables 1 and 2) this type of behaviour could already be analysed in terms of the first order approximation.

In further numerical calculations the influence of numbers of half-waves m for assumed imperfections on the load carrying capacity was neglected ; this simplification is commonly used in the studies known to the present author.

3.2. Open section columns

Geometrical dimensions of column under discussion are shown in Fig. 3.

In thin-walled columns analysed by Ali and Sridharan (1988), with dimensions as follows :

$$b_1/b_2 = 0.5, \quad h_1/h_2 = 1.0, \quad l/b_2 = 78, \quad b_2/h_2 = 50.0, \quad \nu = 0.3,$$

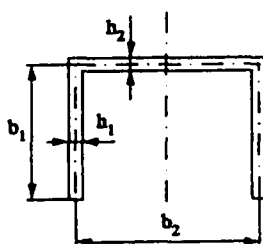


Fig. 3. Open cross-section considered.

Table 3. Load carrying capacity for open column of the following cross-section dimensions:

$$b_1/b_2 = 0.5, \quad h_1/h_2 = 1.0, \quad l/b_2 = 7.8, \quad b_2/h_2 = 50.0$$

$$\text{at imperfections } |\xi_1| = 1.0, |\xi_2| = 0.2, \xi_3 = 0.0$$

		Linear analysis			Nonlinear analysis		
		σ_1^*	σ_2^*	σ_3^*	σ_{s1}^*	Analysis I σ_{s2}^*	Analysis II σ_{s2}^*
1	6	2.508(a)	1.051(s)	1.440(a)	0.851	0.856	1.756
2	6	3.696(s)	1.051(s)	1.440(a)	0.742	0.746	0.994
3	6	3.696(s)	1.051(s)	3.308(s)	0.728	0.731	0.919
4	8	2.508(a)	1.131(s)	1.374(a)	0.857	0.860	1.250
5	8	3.696(s)	1.131(s)	1.374(a)	0.762	0.767	0.982
6	8	3.696(s)	1.131(s)	2.513(s)	0.738	0.741	0.898

the ratio of the global stress value corresponding to the flexural-torsional buckling (the primary global mode) to the stress value of the primary local mode is 2.385, while the ratio of the stress of the global Euler mode (the secondary global mode) to the stress of the primary local mode is 3.545 [in the paper by Ali and Sridharan (1988) these ratios are 2.63 and 4.22, respectively]. Thus taking into account the components: $u_n^{(i)} \neq 0$, $v_n^{(i)} \neq 0$, $w_n^{(i)} \neq 0$ in the first order displacement field causes a decrease in the global stress values [see Kolakowski (1993) for a more detailed analysis].

Table 3 presents nondimensional stresses σ_n^* , the number of half-waves being $m = 6$ and $m = 8$, and the limit load capacity in the first and the second nonlinear approximations (σ_{s1}^* and σ_{s2}^* respectively) for Analysis I and II at the imperfections $|\xi_1| = 1.0$, $|\xi_2| = 0.2$, $\xi_3 = 0.0$ and for some possible combinations of buckling modes. Index n assumes the following values: 1—for the primary or secondary global buckling mode, 2—for the primary local buckling mode, 3—for the secondary local buckling mode. The following code has been used in Table 3 in order to identify the support conditions on the axis of the symmetry of the cross-section: (a)—antisymmetry, (s)—symmetry, respectively for the n th buckling mode.

The case $m = 8$ refers to the minimum value of the secondary local buckling mode.

A more detailed analysis concerning the selection of the buckling mode can be found in papers by Pignataro and Luongo (1987) and Kolakowski (1989a).

The negative value of b_{3333} were obtained for cases 1, 2, 5, 6 as given by Analysis I and for case 3 as given by Analysis II.

Thus, in the case of Analysis I, the behaviour of a column subject to compression can sometimes be properly described already in the first order approximation.

The limit carrying capacity σ_{s2}^* is determined by an interaction of the more dangerous secondary global (Euler's) buckling mode with two symmetric local modes, based on Analysis I at $m = 6$ (case 3) and based on Analysis II at $m = 8$ (case 6).

Table 4 contains calculation results analogous to those from Table 3, with identical imperfection values, but for a column with the following dimensions:

$$b_1/b_2 = 0.3077, \quad h_1/h_2 = 0.8, \quad l/b_2 = 5.0, \quad b_2/h_2 = 52.0, \quad \nu = 0.3.$$

Table 4. Load carrying capacity for open column of the following cross-section dimensions:

$$b_1/b_2 = 0.3077, \quad h_1/h_2 = 0.8, \quad l/b_2 = 5.0, \quad b_2/h_2 = 52.0$$

$$\text{at imperfections } |\xi_1| = 1.0, |\xi_2| = 0.2, \xi_3 = 0.0$$

		Linear analysis			Nonlinear analysis		
		σ_1^*	σ_2^*	σ_3^*	σ_{s1}^*	Analysis I σ_{s2}^*	Analysis II σ_{s2}^*
1	5	2.474(s)	1.289(s)	2.737(a)	1.200	1.738	1.750
2	5	3.541(a)	1.289(s)	2.737(a)	1.121	1.408	1.747
3	5	2.474(s)	1.289(s)	3.828(s)	1.026	1.157	1.185
4	8	2.474(s)	1.612(s)	2.267(a)	1.365	1.502	1.878
5	8	3.541(a)	1.612(s)	2.267(a)	1.243	1.332	1.515
6	8	2.474(s)	1.612(s)	2.726(s)	1.055	1.089	1.110

The primary global buckling mode refers to Euler's mode, while the secondary global mode corresponds to the flexural-torsional one. Also in this case a good separation of global modes from the primary local mode is present. Special attention must be paid to the fact that the coefficients b_{3333} are negative in all cases as given by Analysis I listed in Table 4.

The interaction of the global mode with two local ones whose number of half-waves was $m = 8$, was most dangerous based on Analysis I and II. Similar to the results presented in Table 3, lower values of the limit load were obtained in the case of interaction of the global Euler mode with two local symmetrical modes than in the case of the interaction of a global flexural-torsional mode with local ones. These values do not exceed the dimensionless stresses σ_m^* in the case of the second nonlinear approximation.

Due to a good separation of the global modes from the minimum primary local mode (Tables 3 and 4) this type of behaviour should be analysed in terms of the second order approximation.

In reality, the second order approximation caused a decrease in the range of imperfection sensitivity.

On the basis of the results obtained, one can conclude that the calculations should take into account an interaction of three buckling modes: the global and the most dangerous first and second local (having the same number of half-waves) modes as well as the influence of the values of imperfections on the number of half-waves of the local buckling. Attention should also be paid to the proper selection of local and global (the latter for open columns) buckling modes. This can be accomplished only by means of nonlinear analysis. The minimum values of load carrying capacity σ_{s2}^* taking into account the second order approximation were obtained for Analysis I.

4. CONCLUSIONS

The present paper deals with the analysis of the influence of boundary conditions at the ends of structures on postbuckling behaviour of open- and closed-section thin-walled structures, under compression.

An interaction of three buckling modes was considered in terms of the second order approximation, having continuity conditions at the junctions of the plates.

REFERENCES

- Ali, M. A. and Sridharan, S. (1988). A versatile model for interactive buckling of columns and beam-columns. *Int. J. Solids Structures* **24**, 481–486.
- Bidermann, B. L. (1977). *Mechanics of Thin-walled Structures—Statics*, p. 488. Mashinostroenie, Moscow. (In Russian.)
- Bilstein, W. (1974). Beitrag zur Berechnung vorverformter mit diskreten Längssteifen ausgesteifter, ausschliesslich in Längsrichtung belasteter Rechteckplatten nach der nichtlinearen Beultheorie. *Der Stahlbau* **7/9**, 193–201 and 276–282.
- Byskov, E. and Hutchinson, J. W. (1977). Mode interaction in axially stiffened cylindrical shells. *AIAA JI* **15**, 941–948.
- Klöppel, K. and Bilstein, W. (1971). Ein Verfahren zur Ermittlung der Beullasten beliebiger rechtwinklig abgekanteter offener und geschlossener Profile nach der linearen Beultheorie unter Verwendung eines abgewandelten Reduktionsverfahrens. *Veröffentlichungen des Institutes für Statik und Stahlbau der Technischen Hochschule, Darmstadt*, 16, 160.
- Klöppel, K., Schmied, R. and Schubert, J. (1966). Die Traglast mittig und aussermittig gedrückter dünnwandiger Kastenträger unter Verwendung der nichtlinearen Beultheorie. *Der Stahlbau* **11**, 321–337.
- Koiter, W. T. (1976). General theory of mode interaction in stiffened plate and shell structures. WTHD Report 590, Delft, p. 41.
- Koiter, W. T. and van der Neut, A. (1980). Interaction between local and overall buckling of stiffened compression panels. In *Thin-Walled Structures* (Edited by J. Rhodes and A. G. Walker), part I, pp. 51–56 and part II, pp. 66–86. Granada, St Albans.
- Kolakowski, Z. (1987a). Mode interaction in thin-walled trapezoidal column under uniform compression. *Thin-walled Struct.* **5**, 329–342.
- Kolakowski, Z. (1988). Some aspects of mode interaction in thin-walled stiffened plate under uniform compression. *Engng Trans.* **36**, 167–179.
- Kolakowski, Z. (1989a). Interactive buckling of thin-walled beams with open and closed cross-section. *Engng Trans.* **37**, 375–397.

- Kolakowski, Z. (1989b). Mode interaction in wide plate with angle section longitudinal stiffeners under compression. *Engng Trans.* **37**, 117–135.
- Kolakowski, Z. (1989c). Some thoughts on mode interaction in thin-walled columns under uniform compression. *Thin-walled Struct.* **7**, 23–35.
- Kolakowski, Z. (1993). Interactive buckling of thin-walled beams with open and closed cross-sections. *Thin-Walled Structures*, **15**, 159–183.
- Krolak, M. (ed.) (1990). *Post-buckling Behaviour and Load Carrying Capacity of Thin-walled Plate Girders*, pp. 553. PWN (Polish Scientific Publishers), Warsaw-Lodz. (In Polish.)
- Manevich, A. I. (1981). Interaction of buckling modes of stiffened plate under compression. *Stroitel'naya Mekhanika i Raschet Sooruzhenii*, 24–29. (In Russian.)
- Manevich, A. I. (1982). Theory of interactive buckling of stiffened thin-walled structures. *Prikl'adnaya Matematika i Mekhanika* **46**, 337–345. (In Russian.)
- Manevich, A. I. (1985). Stability of shells and plates with T-section stiffeners. *Stroitel'naya Mekhanika i Raschet Sooruzhenii*, 34–38. (In Russian.)
- Manevich, A. I. (1988). Interactive buckling of stiffened plate under compression. *Mekhanika Tverdogo Tela*, 152–159 (In Russian.)
- Moellmann, H. and Goltermann, P. (1989). Interactive buckling in thin-walled beams. Part I: Theory; Part II: Applications. *Int. J. Solids Structures* **25**, 715–728 and 729–749.
- Pignataro, M., Luongo, A. and Rizzi, N. (1985). On the effect of the local overall interaction on the postbuckling of uniformly compressed channels. *Thin-walled Struct.* **3**, 283–321.
- Pignataro, M. and Luongo, A. (1987). Asymmetric interactive buckling of thin-walled columns with initial imperfection. *Thin-walled Struct.* **3**, 365–386.
- Pignataro, M. and Luongo, A. (1987). Multiple interactive buckling of thin-walled members in compression. *Proc. Int. Colloquium on Stability of Plate and Shell Structures*, Ghent, University Ghent, pp. 235–240.
- Sridharan, S. and Ali, M. A. (1985). Interactive buckling in thin-walled beam-columns. *J. Engng Mech. ASCE* **111**, 1470–1486.
- Sridharan, S. and Ali, M. A. (1986). An improved interactive buckling analysis of thin-walled columns having doubly symmetric sections. *Int. J. Solids Structures* **22**, 429–443.
- Sridharan, S. and Graves-Smith, T. R. (1981). Postbuckling analyses with finite strips. *J. Engng Mech. Div. ASCE* **107**, 869–888.
- Sridharan, S. and Peng, M. H. (1989). Performance of axially compressed stiffened panels. *Int. J. Solids Structures* **25**, 879–899.
- Unger, B. (1969). Elastisches Kippen von beliebig gelagerten und aufgehängten Durchlaufträgern mit ein-fachsymmetrischen, in Trägerachse veränderlichem Querschnitt und einer Abwandlung des Reduktionsverfahrens als Lösungsmethode. Dissertation D17, Darmstadt.

APPENDIX

The coefficients in the nonlinear equilibrium equations (17) a_{ijJ} and b_{ijkJ} are given by the following expressions [see Byskov and Hutchinson (1977) for a more detailed analysis]:

$$a_{ijJ} = [\sigma^{(j)} \cdot I_{11}(U^{(i)}, U^{(j)}) + 2\sigma^{(i)} \cdot I_{11}(U^{(i)}, U^{(j)})] / (2\sigma^{(j)} \cdot \epsilon^{(j)}) \quad (\text{A1})$$

$$b_{ijkJ} = [\sigma^{(j)} \cdot I_{11}(U^{(j)}, U^{(k)}) + \sigma^{(i)} \cdot I_{11}(U^{(k)}, U^{(j)}) + \sigma^{(j)} \cdot I_{11}(U^{(i)}, U^{(jk)}) + \sigma^{(i)} \cdot I_{11}(U^{(i)}, U^{(jk)}) + 2\sigma^{(i)} \cdot I_{11}(U^{(i)}, U^{(kj)})] / (2\sigma^{(j)} \cdot \epsilon^{(j)}) \quad (\text{A2})$$